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Static fluid cylinders and plane layers in general relativity

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Abstract. Static perfect fluid distributions in general relativity which possess cylindrical, toroidal or pseudoplanar symmetry (these symmetries are locally equivalent) are considered. Solutions in quadratures are obtained for fluids with an unspecified equation of state and for $\rho c^2 = np$, where n is a constant, with or without an electromagnetic field $F_{\mu\nu}$ compatible with the symmetry assumed. Moreover, for $\rho c^2 = np$, $F_{\mu\nu} \equiv 0$, solutions are given in an explicit closed form. Some physical properties of the solutions are discussed.

1. Introduction

In this paper static perfect fluid distributions are considered in space–times which are usually called cylindrically symmetric. In a number of papers (see Marder 1958, Teixeira *et al* 1977a, b, Safko and Witten 1972, Krori and Barua 1974 and references therein) solutions of this type have been obtained for special choices of the matter equation of state or for restricted forms of the metric. Evans (1977) reduced the problem for an unspecified equation of state to one second-order ordinary differential equation with two unknown functions; it admits quadratures only for some special choices of one of them. For the equation of state $\rho c^2 = np$, Evans arrived at some first-order equations which can be further integrated only numerically. Here, using another coordinate condition, we solve the problem entirely in quadratures for a perfect fluid with an unspecified equation of state and with $\rho c^2 = np$, with or without an electromagnetic field $F_{\mu\nu}$ compatible with the symmetry considered. The $F_{\mu\nu}$ field may be external with respect to the fluid or may be created by a charge or current distribution within it. The physical properties of these solutions are discussed briefly in § 7; more details will be given elsewhere. For the particular case of charged dust see Bronnikov (1979).

We assume that the space–time is static and possesses two space-like Killing vectors which are orthogonal both to the time-like one and to each other. One can choose the coordinates so that the metric has the form

$$ds^2 = e^{2\gamma(x)}\,dx^{02} - e^{2\lambda(x)}\,dx^2 - e^{2\mu(x)}\,d\eta^2 - e^{2\beta(x)}\,d\xi^2. \tag{1}$$

It corresponds to cylindrical symmetry if trajectories of one of the spatial Killing vectors, i.e. the coordinate lines of (say) ξ, are closed, so that $\xi \in [0, 2\pi)$ and $\eta \in (-\infty, \infty)$ are the azimuthal and longitudinal coordinates respectively. If both ξ and η have closed coordinate lines (hence both of them can be treated as angular coordinates), we can call the system toroidally symmetric. On the contrary, if $\xi \in (-\infty, \infty)$ and

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$\eta \in (-\infty, \infty)$, the symmetry can be called 'pseudoplanar' (to obtain the well known planar symmetry, one should put, in addition, $\beta(x) \equiv \mu(x)$).

Treatment of symmetries such as planar and cylindrical ones allows one to study fields due to isolated bodies with extreme departures from sphericity (discs and rods), avoiding the mathematical difficulties of the general, nonsymmetric case. Planar symmetry is known to be a good approximation for some stages of anisotropic gravitational collapse due to the tendency of stronger contraction of matter along the shorter deformation axis (Binney 1977). Besides, such symmetries are of interest for construction of inhomogeneous cosmological models (see, e.g., Shikin 1972). Pseudoplanar symmetry may correspond to a field of an anisotropic plane layer (with x the thickness coordinate) or, in a certain approximation, a disc formed by the collapse of a triaxial ellipsoid-shaped body. Moreover, planarly or pseudoplanarly symmetric models may be useful for an approximate description of thin layers (or crusts) of massive bodies of any shape. At any rate, one may hope that solutions like those considered in this paper can serve some more realistic purpose than 'filling the gap in the literature' (Evans 1977).

The field equations involve only local quantities and may be considered for all the above symmetries simultaneously.

2. Basic equations

The Einstein tensor G^ν_μ for metric (1) is diagonal. Under the coordinate condition

$$\lambda(x) = \gamma(x) + \mu(x) + \beta(x) \tag{2}$$

it takes the following remarkably symmetric form:

$$\begin{aligned} e^{2\lambda} G^1_1 &= U \overset{\text{def}}{=} \beta'\gamma' + \beta'\mu' + \gamma'\mu' \\ e^{2\lambda} G^0_0 &= \beta'' + \mu'' - U \\ e^{2\lambda} G^2_2 &= \beta'' + \gamma'' - U \\ e^{2\lambda} G^3_3 &= \gamma'' + \mu'' - U \end{aligned} \tag{3}$$

where a prime denotes d/dx. We consider the Einstein equations

$$G^\nu_\mu = -\kappa[T^\nu_\mu(F) + T^\nu_\mu(E)] \qquad \mu, \nu = 0, 1, 2, 3 \tag{4}$$

with $T^\nu_\mu(F)$ the energy–momentum tensor for a static perfect fluid with density ρ and pressure $p(\rho)$.

$$T^\nu_\mu(F) = \mathrm{diag}(\rho c^2, -p, -p, -p), \tag{5}$$

and $T^\nu_\mu(E)$ that for the electromagnetic field $F_{\mu\nu}$. The latter can be (in the cylindrical case) of the three alternative types depending on which components of $F_{\mu\nu}$ survive (see Safko and Witten 1971):

(i) R type, radial electric field, $F_{01} = -F_{10}$;

(ii) L type, longitudinal magnetic field, $F_{13} = -F_{31}$;

(iii) A type, azimuthal magnetic field, $F_{12} = -F_{21}$.

(In each of the three cases arbitrary mixtures of electric and magnetic fields are produced by duality rotations.) In the toroidal and pseudoplanar cases the ξ and η coordinates, and hence the L and A fields, are equivalent. In the special case of planar symmetry a vector quantity with a non-zero component in the (η, ξ) plane cannot exist; thus only the R field is possible.

The Maxwell equations

$$\nabla_\alpha F^{\alpha\mu} = -(4\pi/c)j^\mu \qquad \nabla_\alpha F^{*\alpha\mu} = 0 \tag{6}$$

for the three types of fields yield

$$\{F^{01}, F^{31}, F^{21}\} = 2\mathrm{e}^{-2\lambda}\{Q(x), J_\xi(x), J_\eta(x)\} \tag{7}$$

where in the brackets $\{\}$ only one term for each type of $F_{\mu\nu}$ field should be considered, by the scheme $\{R, L, A\}$. The functions $Q(x)$, $J_\xi(x)$ and $J_\eta(x)$ characterise the integral charge and currents in the ξ and η directions respectively, and are expressed in terms of the four-current density as

$$\{Q(x), J_\xi(x), J_\eta(x)\} = (2\pi/c)\int \mathrm{d}x \cdot \mathrm{e}^{2\lambda}\{j^{\cdot 0}(x), j^{\cdot 3}(x), j^{\cdot 2}(x)\}. \tag{8}$$

To consider the system in an external electromagnetic field one should put just $\{Q, J_\xi, J_\eta\} = \text{constant}$.

In the following we confine ourselves to the R and L fields, noting that solutions for the A field are obtained from those for the L field by merely interchanging the ξ and η coordinates.

The energy–momentum tensors for the R and L fields are

$$T^\nu_\mu(E) = (Q^2/2\pi)\,\mathrm{e}^{2\gamma-2\lambda}\,\mathrm{diag}(1, 1, -1, -1) \tag{9}$$

and

$$T^\nu_\mu(E) = (J_\xi^2/2\pi)\,\mathrm{e}^{2\beta-2\lambda}\,\mathrm{diag}(1, -1, 1, -1) \tag{10}$$

respectively.

The sets of equations to be solved are, in general, indeterminate and the solutions should contain arbitrary functions. In the following we choose them from the functions β, γ, μ and then determine the metric completely using some combinations of the Einstein equations. Other combinations then enable us to find material quantities p, ρ and Q (or J_ξ). When additional relations are introduced (e.g., a concrete form of the equation of state), then, in general, further integration is required.

3. Solution for $F_{\mu\nu} \equiv 0$, $p(\rho)$ unspecified

When the equation of state $p = p(\rho)$ is unspecified, there are four Einstein equations for five unknown functions β, γ, μ, ρ and p (or, equivalently, three independent Einstein equations and the hydrostatic equilibrium condition $p' = -\gamma'(\rho c^2 + p)$). Thus one unknown function may be chosen arbitrarily.

The Einstein equation $\binom{2}{2} - \binom{3}{3}$ yields

$$\beta'' = \mu'' \qquad \mu = \beta + ax + a_1 \tag{11}$$

($a, a_1 = \text{constant}$; changing the scale along, say, the η axis, one can achieve $a_1 = 0$). Moreover, the equation $\binom{1}{1} - \binom{2}{2}$ can be written in two equivalent forms

$$\alpha'' - (2a + 4\beta')\alpha' + 2\beta'^2 = 0 \qquad \alpha \overset{\text{def}}{=} \beta + \gamma \tag{12}$$

$$\alpha'' - 2a\alpha' - 2\alpha'^2 + 4\gamma'^2 = 0 \tag{13}$$

and integrated in quadratures: either (12) as a linear first-order equation in α' with an arbitrary function $\beta(x)$, or (13) as a separable first-order equation in γ with an arbitrary function $\alpha(x)$.

Thus the metric is completely determined. The quantities $p(x)$ and $\rho(x)$ are obtained directly from the equations $\binom{1}{1}+\binom{2}{2}$ and $\binom{1}{1}+\binom{0}{0}$:

$$2\kappa p = \alpha'' \,\mathrm{e}^{-2\lambda} \tag{14}$$

$$\kappa\rho c^2 = \kappa p - 2\beta'' \,\mathrm{e}^{-2\lambda}. \tag{15}$$

The resulting solution depends on one arbitrary function ($\alpha(x)$ or $\beta(x)$) and one essential integration constant a. Note that the solution with planar symmetry corresponds to $a = 0$.

4. Solution for $F_{\mu\nu} \equiv 0$, $\rho c^2 = np$ (n = constant)

Certainly the solution for this equation of state is special with respect to that of § 3. Equation (11) is valid, as before. The set of equations is now determinate and the solution is expressable in terms of elementary functions. Indeed, the equation $2\binom{0}{0}+(n+1)\binom{1}{1}+(n-1)\binom{2}{2}$ gives

$$(n-1)\alpha'' + 4\beta'' = 0 \qquad 4\beta = (1-n)\alpha + 4bx + b_1 \tag{16}$$

(b, b_1 = constant). With (16), equation (12) becomes

$$\alpha'' + A\alpha'^2 - B\alpha' + 2b^2 = 0$$

$$A \overset{\text{def}}{=} \tfrac{1}{8}(n-1)(n+7), \qquad B \overset{\text{def}}{=} 2a + (n+3)b \tag{17}$$

and may be solved easily. Namely, we can write (C_i = constant):

(i) $A = B = 0$:

$$\alpha = -b^2x^2 + C_1x + C_2; \tag{18}$$

(ii) $A = 0, B \neq 0$:

$$\alpha = 2B^{-1}b^2x + C_3 + C_4B^{-2}\,\mathrm{e}^{Bx}; \tag{19}$$

(iii) $A \neq 0, \Delta \overset{\text{def}}{=} B^2 - 8b^2A$:

$$\exp(A\alpha - \tfrac{1}{2}Bx) = \begin{cases} C_5 \exp(\frac{1}{2}\Delta^{1/2}x)/\exp(-\frac{1}{2}\Delta^{1/2}x) & \text{for } \Delta > 0 \\ C_7x + C_8 & \text{for } \Delta = 0 \\ C_9 \cos[\frac{1}{2}(-\Delta)^{1/2}x + C_{10}] & \text{for } \Delta < 0. \end{cases} \tag{20}$$

Note that $A > 0$ for the physically plausible choice $n > 1$; $n = 1$ implies $A = 0$. The quantities $p(x)$ and $\rho(x)$ are found from equations (14) and (15) as before. The solution contains two essential integration constants a and b; the others (a_1, b_1 and C_i) can be eliminated by proper choices of scales along the x^0, η and ξ axes and the origin of the x coordinate.

5. Solutions for $F_{\mu\nu} \neq 0$, $p(\rho)$ unspecified

5.1. *The R field*

The equation $\binom{2}{2} - \binom{3}{3}$ yields (11) expressing $\mu(x)$ in terms of $\beta(x)$. Taking $\alpha(x) = \beta + \gamma$ and $\beta(x)$ as arbitrary functions, we can express the other unknown functions p, Q and ρ by the equations $\binom{1}{1} + \binom{2}{2}$, $\binom{1}{1} - \binom{2}{2}$ and $2\binom{0}{0} - \binom{1}{1} + \binom{2}{2}$ respectively. This results in (14) and

$$\kappa Q^2(x)\, e^{2\gamma}/\pi = \alpha'' - 2U, \qquad U = 2\alpha'\beta' + a\alpha' - \beta'^2 \tag{21}$$

$$2\kappa\rho c^2\, e^{2\lambda} = 4U - 4\beta'' - \alpha''. \tag{22}$$

The two functions $\alpha(x)$ and $\beta(x)$ stand for arbitrariness in the fluid equation of state and the charge distribution.

5.2. *The L field*

The equation $\binom{1}{1} - \binom{3}{3}$ gives

$$2\beta'(\mu' + \gamma') = \mu'' + \gamma'' - 2\mu'\gamma. \tag{23}$$

For the case $\mu' + \gamma' = 0$ we get a solution with $\rho c^2 + p = 0$ and one arbitrary function $\beta(x)$:

$$\mu = \gamma = 0, \qquad -p = \rho c^2 = J_\xi/2\pi \overset{\text{def}}{=} \Lambda(x)/\kappa; \qquad -2\Lambda = \beta''\, e^{-2\beta}.$$

for constant Λ,

$$e^\beta = (2\Lambda)^{-1/2} k^{-1} \cosh k(x - x_0) \qquad k_1 x_0 = \text{constant}; \tag{24}$$

this is a special electrovacuum solution with a cosmological constant Λ.

For the case $\mu' + \gamma' \neq 0$ we can take $\mu(x)$ and $\gamma(x)$ as arbitrary functions. Consequently, equation (23) gives $\beta(x)$ as a quadrature,

$$2\beta = \log|\gamma' + \mu'| - 2 \int \mu'\gamma'(\mu' + \gamma')^{-1}\, dx \tag{25}$$

and the quantities $p(x)$, $J_\xi(x)$ and $\rho(x)$ are easily found from the equations $\binom{1}{1} + \binom{2}{2}$, $\binom{2}{2} - \binom{3}{3}$ and $2\binom{0}{0} + \binom{1}{1} - \binom{2}{2}$ respectively. This results in (14) (remember that $\alpha = \beta + \gamma$) and

$$\kappa J_\xi^2\, e^{2\beta}/\pi = \mu'' - \beta'' \tag{26}$$

$$2\kappa\rho c^2\, e^{2\lambda} = \gamma'' - \beta'' - 2\mu''. \tag{27}$$

The two functions $\mu(x)$ and $\gamma(x)$ stand for arbitrariness in the equation of state and the current distribution.

6. Solutions for $F_{\mu\nu} \neq 0$ and $\rho c^2 = np$

6.1. *The R field*

The solution is special with respect to (11), (14), (21) and (22). In particular, equation (11) is valid and the equation $2\binom{0}{0} + (n-1)\binom{1}{1} + (n+1)\binom{2}{2}$ leads to

$$\tfrac{1}{4}(n+1)\alpha'' - (2\beta' + a)\alpha' + \beta'' + \beta'^2 = 0. \tag{28}$$

This is a linear first-order equation in $\alpha'(x)$ if we take $\beta(x)$ as an arbitrary function.

Hence the problem is reduced to quadratures. The quantities $p(x)$ and $Q(x)$ can be found, as before, from equations (14) and (21).

6.2. The L field

The solution is special with respect to (14) and (25)–(27). In particular, equation (23) is valid. Also, the equation $2\binom{0}{0}+(n+1)\binom{1}{1}+(n-1)\binom{2}{2}$ gives

$$(n+1)\beta+(n-1)\gamma+2\mu = ax+a_1 \tag{29}$$

(a, a_1 = constant; a_1 is inessential as before) and we can substitute β from (29) into (23). The resulting equation takes an integrable form if we choose $\nu(x)=\mu+\gamma$ as an arbitrary function and $\zeta(x)=(n-1)\gamma-2\mu$ ($n\neq-1$, see (24)) as an unknown function:

$$2\zeta'^2+(n+1)^2\nu''-2a(n+1)\nu'+4(n-1)\nu'^2=0. \tag{30}$$

Equations (29) and (30) determine the metric; $p(x)$ and $J_\xi(x)$ are again determined from (26) and (27).

The arbitrary functions of this section correspond to charge and current distributions.

7. Discussion

The above solutions are obtained under very general conditions. They are physically resonable if the arbitrary functions and constants satisfy some additional requirements; some of these are discussed here.

1. When the fluid equation of state is unspecified, the function $p=p(\rho)$ is known in a parametric form: $p=p(x)$, $\rho=\rho(x)$. A physically reasonable solution should be such that

$$0\leqslant p\leqslant \rho c^2 \tag{31}$$

$$0\leqslant \mathrm{d}p/\mathrm{d}\rho\leqslant c^2. \tag{32}$$

The corresponding limitations upon the arbitrary functions are obtained in a straightforward way from the solutions. For example, for the solution of § 3, (31) implies $\alpha''\geqslant 0$, $\beta''\leqslant 0$ (if $\alpha(x)$ is arbitrary, then $\beta(x)$ is expressed in terms of α and $\beta''\leqslant 0$ is an implicit limitation upon $\alpha(x)$ and vice versa). Requirement (32) upon the velocity of sound $(\mathrm{d}p/\mathrm{d}\rho)^{1/2}$ leads to complicated conditions involving third-order derivatives.

2. For cylindrically symmetric configurations it is natural to require that the solution should be regular at an axis if it exists (i.e., if $e^\beta=0$ at some $x=x_a$). This means that the space–time should be locally flat and p, ρ and $F_{\mu\nu}$ have no singularities at $x=x_a$; in particular,

$$|\gamma|<\infty \qquad |\mu|<\infty \qquad \mathrm{e}^{-\lambda}\gamma'\to 0 \qquad \mathrm{e}^{2\beta-2\lambda}\beta'^2\to 1. \tag{33}$$

The third condition means that the gravitational force acting upon a static test particle vanishes at the axis; the fourth one requires a proper circumference–radius ratio. For all the solutions with $F_{\mu\nu}\equiv 0$ and with the R field, this implies, in particular, $a\neq 0$ and $x_a=\pm\infty$ for $a\gtrless 0$ (see (11)).

3. For pseudoplanarly symmetric configurations it is of interest to know under what conditions the fluid layer is regular and symmetric under reflections with respect to a certain plane, say $x=0$. This should be the case if the solution approximates the field of

a finite disc which is symmetric in the thickness direction. This requirement means that the functions $\rho, p, \gamma, \mu, \beta$ should be even $(f(-x) = f(x))$. For all the solutions with $F_{\mu\nu} \equiv 0$ and with the R field, this implies $a = 0$ in equation (11); thus the symmetry is necessarily planar. Amazingly, a discrete symmetry requirement for the x direction creates additional symmetry in the (η, ξ) plane!

4. The explicit form of the solutions of § 4 confirms Evans' (1977) statement on the 'instability in the equation of state' for $n \to 1$. This means there is no smooth transition from solutions with $n > 1$ to the solution with $n = 1$. Other 'instabilities' of this kind, concerning smoothness in the integration constants a and b, are easily observed in equation (18)–(20).

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